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CBS constants for multilevel splitting of graph-Laplacian and application to preconditioning of discontinuous Galerkin systems

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Dedicated to Professor Henryk Wozniakowski on the occasion of his 60th birthday

Abstract

The goal of this work is to derive and justify a multilevel preconditioner of optimal arithmetic complexity for symmetric interior penalty discontinuous Galerkin finite element approximations of second order elliptic problems. Our approach is based on the following simple idea given in [R.D. Lazarov, P.S. Vassilevski, L.T. Zikatanov, Multilevel preconditioning of second order elliptic discontinuous Galerkin problems, Preprint, 2005]. The finite element space $\mathscr{V}$ of piece-wise polynomials, discontinuous on the partition $\mathscr{T}$, is projected onto the space of piece-wise constant functions on the same partition that constitutes the largest space in the multilevel method. The discontinuous Galerkin finite element system on this space is associated to the so-called "graph-Laplacian". In 2-D this is a sparse M-matrix with -1 as off diagonal entries and nonnegative row sums. Under the assumption that the finest partition is a result of multilevel refinement of a given coarse mesh, we develop the concept of hierarchical splitting of the unknowns. Then using local analysis we derive estimates for the constants in the strengthened Cauchy–Bunyakowski–Schwarz (CBS) inequality, which are uniform with respect to the levels. This measure of the angle between the spaces of the splitting was used by Axelsson and Vassilevski in [Algebraic multilevel preconditioning methods II, SIAM J. Numer. Anal. 27 (1990) 1569–1590] to construct an algebraic multilevel iteration (AMLI) for finite element systems. The main contribution in this paper is a construction of a splitting that produces new estimates for the CBS

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constant for graph-Laplacian. As a result we have a preconditioner for the system of the discontinuous Galerkin finite element method of optimal arithmetic complexity.

Keywords: Discontinuous Galerkin; Second order elliptic equation; Graph-Laplacian; Multilevel preconditioning; CBS constant

1. Introduction

Consider a second order elliptic problem on a polygonal domain $\Omega \subset R^2$:

$$\begin{aligned} -\nabla \cdot (a(x)\nabla u) &= f(x) \quad \text{in } \Omega, \\ u(x) &= g_D \quad \text{on } \Gamma_D, \\ a\nabla u \cdot \mathbf{n} &= g_N \quad \text{on } \Gamma_N. \end{aligned} \tag{1.1}$$

Here $\mathbf{n}$ is the exterior unit normal vector to $\partial\Omega \equiv \Gamma$. The boundary is assumed to be decomposed into two disjoint parts Γ_D and Γ_N, $\Gamma_D \cap \Gamma_N = \emptyset$ and the boundary data g_D, g_N are smooth. For the formulation below we shall need the existence of the traces of u and $a\nabla u \cdot \mathbf{n}$ on certain interfaces in Ω. Thus, the solution u is assumed to have the required regularity. To simplify our exposition we assume that the set Γ_D is not empty and its 1-D measure is nonzero.

In Section 2 we introduce two discontinuous Galerkin FEMs for the above problem that lead to symmetric and positive definite algebraic problems. These are special cases of a more general class of discontinuous Galerkin schemes for second order elliptic problems (see, e.g. [1,10,13,17]).

Below we comment on some of the first works on efficient solution methods for system linear equations arising in discontinuous Galerkin approximations [9,15,16].

In [15] Gopalakrishnan and Kanschat discuss and study multigrid method (MG) for a symmetric discontinuous Galerkin method. A multigrid with variable number of smoothing steps is considered under the standard assumptions: (1) there is a sequence of nested triangulations $\mathcal{T}_1 \subset \mathcal{T}_2 \subset \cdots \subset \mathcal{T}_J$ and multilevel spaces $\mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \subset \mathcal{V}_J$ of piece-wise polynomials that define a sequence of operators $A_1, A_2, \ldots, A_J$ and corresponding projections P_{k-1} : $\mathcal{V}_k \mapsto \mathcal{V}_{k-1}$; (2) P_{k-1} has certain weak approximation property: for all $u \in \mathcal{V}_k$, $k = 2, \ldots, J$: $|||u - P_{k-1}u|||_k \leqslant Ch_k\|A_k u\|_{-1+\beta}$. It is shown that if the number $m(k)$ of smoothing steps increases as k decreases, say $m(k) = 2^{J-k}$, then MG preconditioner has optimal arithmetic complexity comparable to the complexity of W-cycle.

Brenner and Zhao [9] considered rectangular partitions and bilinear finite element spaces in the discontinuous Galerkin method for the above problem. Their main result for V-cycle could be summarized as follows: if the solution satisfies the a priori estimate $\|u\|_{H^{1+\alpha}} \leqslant C\|f\|_{H^{-1+\alpha}}$, $\alpha \in (\frac{1}{2}, 1]$, then there is m_0 which is independent of the number of levels k such that the norm of the multigrid error propagation operator E_{mg} satisfies the estimate $\left|\left|\left|E_{\mathrm{mg}}\right|\right|\right| \leqslant C/m^{\alpha},\ k \geqslant 1,\ m \geqslant m_0$. This shows that the higher smoothness of the solution improves the MG convergence. Similar results are obtained for the W-cycle.

Recently in [11,16] two-level and multilevel iteration methods for solving discontinuous Galerkin systems have been proposed and studied. The approach is based on the following idea: first apply the classical two-grid method involving the original space of discontinuous functions $\mathcal{V}$ and an auxiliary space $\mathcal{V}^{(0)}$ of piece-wise polynomials defined on the same mesh and then use the algebraic multilevel iteration (AMLI) of Axelsson and Vassilevski [3,18]. In [11,16] three different choices for the auxiliary space $\mathcal{V}^{(0)}$ have been proposed and studied: (1) continuous linear finite elements, (2) nonconforming Crouzeix–Raviart elements, (3) space of discontinuous piece-wise constant functions. Then the convergence of the two-grid method is considered in the general framework established in [14]. Under certain assumptions it has been proved that the two-grid method converges independently of the mesh size. The numerical results presented in [11] well illustrate the robust scalability of the algorithm. In [16] a multilevel extension based on the AMLI of Axelsson and Vassilevski [3,18] has been studied.

The third choice of spaces led to a novel and interesting problem from mathematical viewpoint. The discontinuous Galerkin scheme on the space $\mathcal{V}^{(0)}$ of discontinuous piece-wise constant functions produces a symmetric and positive definite matrix, called "graph-Laplacian". In the past such algebraic problems were generated by cell-centered approximations of elliptic equations on rectangular grids. On rectangular grids the "graph-Laplacian" approximates the Laplacian and the analysis could use some of the tools of the general multigrid theory. Multigrid preconditioners of optimal complexity for such problems were analyzed and tested in [8]. On an irregular grid the corresponding linear problem does not approximate an elliptic problem and the study of optimal preconditioners does not fit into the general framework of multigrid or multilevel methods.

In this paper we consider AMLI method for preconditioning the graph-Laplacian matrix. The AMLI method is suitable for such situations since it could be analyzed by algebraic means, see e.g. [3,5,6,12,18]. We have assumed that the matrix is generated from a finite element partition obtained by a regular refinement of a given initial mesh consisting of both triangles and quadrilaterals. We note that this approach is applicable to more general partitions, including pentagons, etc. and is further capable of constructing and analyzing AMLI preconditioners for discontinuous Galerkin systems in 3-D.

One way to construct and justify optimal preconditioners is to introduce multilevel splitting of the unknowns as proposed originally by Bank and Dupont in [4] for the standard finite element approximations. Our study is based on certain properties of the hierarchical splitting of the spaces of piece-wise constant functions represented by their nodal bases. These include the locality of the new basis, so that the pivot block in the two-level matrix has a uniformly bounded condition number. This block corresponds to the unknowns which are complementary to the coarse grid unknowns. The related second diagonal block can be viewed as a certain aggregation of the current two-level matrix. Within a suitably introduced parametric setting, we require this block to be not only associated but equal to the coarse grid matrix. The key role in the derivation of optimal convergence rate estimates plays the constant γ in the strengthened CBS inequality, associated with the angle between the two subspaces of the hierarchical splitting.

It turns out that existence only of a uniform estimate for this constant is not enough, therefore, accurate quantitative bounds for γ have to be found as well. More precisely, the value of the upper bound for $\gamma \in (0, 1)$ is a part of the construction of the multilevel extension of the related two-level method. Thus, the main contribution of our paper is construction of a splitting of the piece-wise constant spaces generated by hierarchy of partitions of triangles and quadrilaterals that produced new estimate for Cauchy–Bunyakowski–Schwarz (CBS) constant for graph-Laplacian. This in turn has generated an optimal AMLI preconditioner for the graph-Laplacian and therefore for the discontinuous Galerkin systems as well.

The paper is organized as follows. In Section 2 we describe two symmetric interior penalty discontinuous Galerkin (IPDG) finite element approximations of second order elliptic problem. The two-grid algorithm which reduces the problem to a system with graph-Laplacian is introduced in Section 3. Sections 4 and 5 contain the needed setting of the AMLI method and the theory of the CBS constant. The new estimates of the CBS constant for graph-Laplacian and the related optimal multilevel preconditioner are presented in Section 6.

2. Discontinuous Galerkin FE approximation

Let $\mathcal{T}$ be a partitioning of Ω into finite number of open subdomains (finite elements) K with boundaries ∂K. We assume that the partition is quasi-uniform and regular. For each finite element we denote by h_K its size (say its diameter) and further $h = \max_{K\in\mathcal{T}} h_K$. Let $e = \overline{K}_1 \cap \overline{K}_2$ be the interface (or edge) of two adjacent subdomains K_1, K_2. The set of all such interfaces is denoted by $\mathcal{E}_0$, note that these edges are inside Ω. Further, $\mathcal{E}_D$ and $\mathcal{E}_N$ will be the edges of finite elements on the boundary Γ_D and Γ_N, respectively. Finally, $\mathcal{E}$ will be the set of all edges: $\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}_D \cup \mathcal{E}_N$. Here we allow finite elements of polygonal shape with hanging nodes, etc. The important assumption is that if e is an edge of a finite element $K \in \mathcal{T}$ then $|e| \approx h_K$. In other words we do not allow very small edges.

On the partition $\mathcal{T}$ we define the finite element space

$$\mathcal{V} := \mathcal{V}(\mathcal{T}) := \{v \in L^2(\Omega) : v|_K \in P_r(K),\ K \in \mathcal{T}\},$$

where P_r is the set of polynomials of degree $r \geqslant 0$.

For each $e = \bar{K} \cap \bar{K}' \in \mathcal{E}_0$ we define the jump $[\![v]\!]$ of any function $v \in \mathcal{V}$ as the vector

$$[\![v]\!]_e := \begin{cases} v|_K \cdot \mathbf{n} + v|_{K'} \cdot \mathbf{n}', & e = \bar{K} \cap \bar{K}' \text{ i.e. } e \in \mathcal{E}_0, \\ v|_K \cdot \mathbf{n}, & e = \bar{K} \cap \Gamma_D \text{ i.e. } e \in \mathcal{E} \setminus \mathcal{E}_0. \end{cases}$$

Here $\mathbf{n}$ and $\mathbf{n}'$ are the external unit vectors to K and K', respectively.

We shall also need the following notation for the average value of the traces of the normal component of a vector function $\mathbf{v} \in \mathcal{V}$ on $e = \bar{K} \cap \bar{K}'$:

$$\{\mathbf{v}\}|_e := \begin{cases} \frac{1}{2}\{\mathbf{v}|_K \cdot \mathbf{n} - \mathbf{v}|_{K'} \cdot \mathbf{n}'\}, & e = \bar{K} \cap \bar{K}' \text{ i.e. } e \in \mathcal{E}_0, \\ \mathbf{v}|_K \cdot \mathbf{n}, & e = \bar{K} \cap \Gamma_D \text{ i.e. } e \in \mathcal{E} \setminus \mathcal{E}_0, \end{cases}$$

and the piece-wise constant function $h_{\mathcal{E}}$ defined on $\mathcal{E}$ as

$$h_{\mathcal{E}} = h_{\mathcal{E}}(x) = |e| \quad \text{for } x \in e \in \mathcal{E}.$$

Further denote

$$(a\nabla v, \nabla v)_{\mathcal{T}} := \sum_{K\in\mathcal{T}} \int_K a\nabla u\ \nabla v\, dx,$$

$$\left\langle h_{\mathcal{E}}^{-1}\, [\![u]\!],\ [\![v]\!] \right\rangle_{\mathcal{E}\cup\mathcal{E}_D} := \sum_{e\in\mathcal{E}\cup\mathcal{E}_D} \int_e h_{\mathcal{E}}^{-1}\, [\![u]\!] \cdot [\![v]\!]\, ds.$$

Finally, we shall use the following norm on $\mathcal{V}$:

$$|\!|\!| v |\!|\!|_h^2 = (a\nabla v, \nabla v)_{\mathcal{T}} + \kappa \left\langle h_{\mathcal{E}}^{-1}\, [\![v]\!],\ [\![v]\!] \right\rangle_{\mathcal{E}\cup\mathcal{E}_D}. \tag{2.1}$$

We shall consider the following symmetric IPDG finite element method (see, e.g. [1]): find $u_h \in \mathcal{V}$ such that

$$\mathcal{A}(u_h, v) = \mathcal{L}(v) \quad \forall\, v \in \mathcal{V}, \tag{2.2}$$

where

$$\begin{aligned}\mathcal{A}(u_h, v) \equiv {}& (a\nabla u_h, \nabla v)_{\mathcal{T}} + \kappa \left\langle h_{\mathcal{E}}{}^{-1} [\![u_h]\!],\ [\![v]\!] \right\rangle_{\mathcal{E}\cup\mathcal{E}_D} \\ & - \langle \{a\nabla u_h\},\ [\![v]\!]\rangle_{\mathcal{E}\cup\mathcal{E}_D} - \langle [\![u_h]\!],\ \{a\nabla v\}\rangle_{\mathcal{E}\cup\mathcal{E}_D}\end{aligned} \tag{2.3}$$

and

$$\mathcal{L}(v) = (f, v) + \langle g_N,\ v\rangle_{\mathcal{E}_N} - \langle g_D,\ a\nabla v\cdot \mathbf{n}\rangle_{\mathcal{E}_D} + \kappa \left\langle h_{\mathcal{E}}{}^{-1} g_D,\ v\right\rangle_{\mathcal{E}_D}. \tag{2.4}$$

It is well known that if κ is sufficiently large then the bilinear form (2.3) is coercive and bounded in $\mathcal{V}$ equipped with norm (2.1) (see, e.g. [1]).

Another symmetric discontinuous Galerkin scheme could be derived by using an approach developed in the work of Ewing et al. [13]. In this case we get a bilinear form

$$\begin{aligned}\mathcal{A}(u_h, v) \equiv {}& (a\nabla u_h, \nabla v)_{\mathcal{T}} + \kappa \left\langle h_{\mathcal{E}}{}^{-1} [\![u_h]\!],\ [\![v]\!] \right\rangle_{\mathcal{E}\cup\mathcal{E}_D} \\ & - \langle \{a\nabla u_h\},\ [\![v]\!]\rangle_{\mathcal{E}\cup\mathcal{E}_D} - \langle [\![u_h]\!],\ \{a\nabla v\}\rangle_{\mathcal{E}\cup\mathcal{E}_D} \\ & - \tfrac{1}{4}\kappa^{-1} \langle h_{\mathcal{E}}\, [\![a\nabla u_h\cdot\mathbf{n}]\!],\ [\![a\nabla v\cdot \mathbf{n}]\!]\rangle_{\mathcal{E}_0}\end{aligned} \tag{2.5}$$

which is coercive for sufficiently large κ. Note that the corresponding DG scheme is slightly different from (2.2).

We summarize the main results regarding the discontinuous Galerkin method (2.2) in the following lemma:

Lemma 2.1. *Assume that the finite element partition $\mathcal{T}$ is regular and locally quasi-uniform. Then the bilinear form $\mathcal{A}(\cdot,\cdot)$ defined by* (2.3) *or* (2.5) *is coercive and bounded in $\mathcal{V}$ equipped with the norm* (2.1) *for any sufficiently large $\kappa > 0$ and the discontinuous Galerkin method* (2.2) *has unique solution.*

3. Two-level method

Now we present the two-level (also called two-grid) iteration method, which for DG systems was introduced in an algebraic setting in [11,16] and studied in the general algebraic framework of [14]. Together with the DG space $\mathcal{V}$ it uses an auxiliary, in general smaller, space $\mathcal{V}^{(0)}$ and proper restriction and prolongation operators. In [16] three possibilities for $\mathcal{V}^{(0)}$ were considered and studied. Here, we take one of these, namely $\mathcal{V}^{(0)}$, as the space of piece-wise constant functions over the partition $\mathcal{T}$. To describe the two-grid method and the results from [11] we need some matrix notations. We shall first reformulate problem (2.2) in terms of matrices and the vector spaces of degrees of freedom. Let n and n_0 be the dimensions of the spaces $\mathcal{V}$ and $\mathcal{V}^{(0)}$, respectively, and let $\{\varphi_j\}_{j=1}^{n}$ and $\{\chi_j\}_{j=1}^{n_0}$ be their nodal bases. In the case we consider here, χ_j is the characteristic function of the finite element $K_j \in \mathcal{T}$ and n_0 is the number of finite elements in $\mathcal{T}$. We denote by $\mathbb{V}$ and $\mathbb{V}^{(0)}$ the spaces of n- and n_0-dimensional vectors of the degrees of freedom of $\mathcal{V}$ and $\mathcal{V}^{(0)}$, respectively. These could be identified as the spaces $\mathbb{R}^n$ and $\mathbb{R}^{n_0}$. For linear finite elements over triangular mesh the space $\mathbb{V}$ is identified with $\mathbb{R}^{3n_0}$ (i.e. $n = 3n_0$), for bilinear elements over quadrilateral mesh, with $\mathbb{R}^{4n_0}$ (i.e. $n = 4n_0$), while for both cases $\mathbb{V}^{(0)}$ is identified with $\mathbb{R}^{n_0}$. The elements of $\mathbb{V}$ and $\mathbb{V}^{(0)}$ are further denoted in bold face, i.e. $\mathbf{u}$, $\mathbf{v}$, etc.

Each coarse grid basis function $\chi_k \in \mathcal{V}^{(0)}$ has unique expansion via the basis of $\mathcal{V}$:

$$\chi_k = \sum_{i=1}^{n} p_{jk}\varphi_j, \quad k = 1, \dots, n_0. \tag{3.1}$$

Now we introduce the matrix $P_0 = \{p_{ik}\}$, $j = 1, \dots, n$ and $k = 1, \dots, n_0$, which could be viewed as an injection (prolongation) operator from $\mathbb{V}^{(0)}$ to $\mathbb{V}$.

To avoid proliferation of indexes further we shall leave out the subindex h in our notation, that is, u_h is replaced by u.

We define the standard ℓ_2-inner product for elements on $\mathbb{V}$ and $\mathbb{V}^{(0)}$:

$$(u, v)_{\ell_2} = \mathbf{v}^T\mathbf{u} \quad \text{for } \mathbf{v}, \mathbf{u} \in \mathbb{V} \text{ (or } \mathbf{v}, \mathbf{u} \in \mathbb{V}^{(0)}).$$

Then we introduce the matrix $\tilde{A} = \tilde{A}_D + \tilde{A}_P$, where

$$(\tilde{A}_D\mathbf{u}, \mathbf{v})_{\ell_2} = (a\nabla u, \nabla v)_{\mathcal{T}}, \quad (\tilde{A}_P\mathbf{u}, \mathbf{v})_{\ell_2} = \kappa\left\langle h_{\mathcal{E}}^{-1} [\![u]\!],\ [\![v]\!]\right\rangle_{\mathcal{E}\cup\mathcal{E}_D}.$$

Obviously both matrices are symmetric and $\tilde{A}_D$ is semidefinite, while $\tilde{A}_P$ is positive definite. Next we define the "stiffness matrices" A and $A^{(0)}$ by the identities

$$(A\mathbf{u}, \mathbf{v})_{\ell_2} = \mathcal{A}(u, v), \quad \mathbf{v}, \mathbf{u} \in \mathbb{V}, \quad (A^{(0)}\mathbf{u}, \mathbf{v})_{\ell_2} = \mathcal{A}(u, v), \quad \mathbf{v}, \mathbf{u} \in \mathbb{V}^{(0)}.$$

Because of expansion (3.1), obviously, $A_0 = P_0^T A P_0$. Finally, we introduce the norm

$$|||v|||^2 = |||\mathbf{v}|||^2 = (\tilde{A}\mathbf{v}, \mathbf{v})_{\ell_2} = (\tilde{A}_D\mathbf{v}, \mathbf{v})_{\ell_2} + (\tilde{A}_P\mathbf{v}, \mathbf{v})_{\ell_2},$$

where $u, v \in \mathcal{V}$ and by duality $\mathbf{u}, \mathbf{v} \in \mathbb{V}$.

From the symmetry, the coercivity and the boundedness of the bilinear form $\mathcal{A}$ it follows that A is symmetric and positive definite matrix that is spectrally equivalent to the matrix $\tilde{A}$. For studying the convergence of the two-grid iteration we shall also need the operator norm $|||A|||$. Let M be a smoothing matrix that satisfies the condition $M^T + M - A$ is symmetric and positive definite. The following two-level method has been studies and justified in [14]:

Two-level algorithm.

(0) *Let* $\mathbf{u}_0$ *be given*
For $\mathbf{u}_i$ "*approximating*" $\mathbf{u}$, *the solution of* $A\mathbf{u} = \mathbf{b}$, *define* $\mathbf{u}_{i+1}$ *as follows*:
(1) *Set* $\mathbf{x}_1 = \mathbf{u}_i - M^{-1}(A\mathbf{u}_i - \mathbf{b})$ (*presmooth*)
(2) $\mathbf{x}_2 = \mathbf{x}_1 - P_0(A^{(0)})^{-1}P_0^T(A\mathbf{x}_1 - \mathbf{b})$ (*correct*)
(3) $\mathbf{u}_{i+1} = \mathbf{x}_2 - M^{-T}(A\mathbf{x}_2 - \mathbf{b})$ (*postsmooth*).

More general two-grid methods with m presmoothing and m postsmoothing steps could be also justified.

In [14], the convergence of the two-level method, which is characterized by the error transfer operator E_{tg}, that is, $\mathbf{u} - \mathbf{u}_{i+1} = E_{tg}(\mathbf{u} - \mathbf{u}_i)$, has been established in the following form:

$$\left|\left|\left|E_{tg}\right|\right|\right| = 1 - 1/K, \quad K \leqslant \sup_{\mathbf{v}\in\mathbb{V}} \frac{\|I - Q\mathbf{v}\|_{\ell_2}^2}{|||\mathbf{v}|||^2},$$

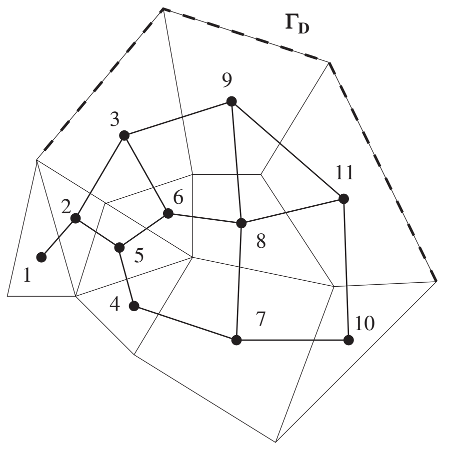

Fig. 1. Partition $\mathcal{T}$ and related graph-Laplacian.

where $Q : \mathbb{V} \mapsto \mathbb{V}^{(0)}$ is an ℓ_2-orthogonal projection operator. A sufficient condition for convergence, independent of the step size, is existence of an operator $Q : \mathbb{V} \mapsto \mathbb{V}^{(0)}$ such that

$$\|(I - Q)\mathbf{v}\|^2_{\ell_2} \leqslant C\, |||\mathbf{v}|||^2, \quad \forall \mathbf{v} \in \mathbb{V}.$$

In [11] the following result has been obtained by using the general theory of [14]:

Theorem 3.1. *The two-level method with Gauss–Seidel as a smoother and coarse space $\mathcal{V}^{(0)}$ of piece-wise constant functions is uniformly convergent with respect to the number of degrees of freedom.*

Further, in [16] this result has been extended to a multilevel method using the general framework of the AMLI of Axelsson and Vassilevski [3] and the basic properties of the two-level projection methods of Falgout et al. [14]. Our goal is to obtain similar results by using multilevel splitting of the unknowns and establishing sharp estimate for the angle between the corresponding spaces.

Now consider the bilinear form $\mathcal{A}(\cdot, \cdot)$, defined by (2.3) or (2.5), on the space $\mathcal{V}^{(0)}$ of piece-wise constant functions, which reduces the formula to the jump part only, $\kappa \left\langle h_{\mathcal{E}}^{-1} [\![u]\!],\ [\![v]\!] \right\rangle_{\mathcal{E}\cup\mathcal{E}_D}$. Then $A^{(0)}$, further called "graph-Laplacian" is defined by $(A^{(0)}\mathbf{u}, \mathbf{v})_{\ell_2} = \left\langle h_{\mathcal{E}}^{-1} [\![u]\!],\ [\![v]\!] \right\rangle_{\mathcal{E}\cup\mathcal{E}_D}$ for $u, v \in \mathcal{V}^{(0)}$. Now we associate the partition $\mathcal{T}$ with a planar graph. The finite elements are the vertices and the interfaces of the finite elements are the edges of the graph. Then taking as degrees of freedom the values of a function in $\mathcal{V}^{(0)}$ over each finite element, we shall get a matrix that has an entry -1 at the s graph vertices connected to a chosen graph vertex and an entry s at the vertex itself. As an illustration the matrix representing "graph-Laplacian" for a particular mesh is given in Fig. 1. For any partitions into quadrilaterals, regardless of the shape, we get the standard 4-point stencil with $4, -1, -1, -1, -1$, probably the reason for the name.

We note that the case of piece-wise constant space is simple and natural. It is a generalization of the technique of cell-centered schemes that are still popular and frequently exploited in petroleum reservoir modeling using rectangular (or parallelepiped) meshes. These schemes are produced either by finite difference approximation of the elliptic problem or by mixed finite element approximations with subsequent elimination of the fluxes. Preconditioners for such systems were developed in [8]. Important ingredient of the analysis in [8] is the fact that on an orthogonal grid the cell-centered scheme has approximation property on all levels. The matrix of the graph-Laplacian is a symmetric M-matrix. However, this matrix does not have any approximation property on an arbitrary grid. Therefore, the multigrid theory that relies on such property (see, e.g. [8]) cannot be used for designing a robust preconditioner by using "graph-Laplacian".

Below is the matrix of graph-Laplacian for the mesh shown in Fig. 1:

$$A^{(0)} = \begin{bmatrix} 1 & -1 & & & & & & & & & \\ -1 & 3 & -1 & & -1 & & & & & & \\ & -1 & 4 & & & -1 & & & -1 & & \\ & & & 2 & -1 & & -1 & & & & \\ & -1 & & -1 & 3 & -1 & & & & & \\ & & -1 & & -1 & 3 & & -1 & & & \\ & & & -1 & & & 3 & -1 & & -1 & \\ & & & & & -1 & -1 & 4 & -1 & & -1 \\ & & -1 & & & & & -1 & 4 & & -1 \\ & & & & & & -1 & & & 2 & -1 \\ & & & & & & & -1 & -1 & -1 & 4 \end{bmatrix}.$$

4. AMLI preconditioner

We construct hierarchical basis functions (HB) for multilevel preconditioners for the algebraic system involving graph-Laplacian. To this end we follow the framework for constructing HB two-level preconditioners for conforming FEM, as described e.g. in [4], and their multilevel extensions, known as AMLI, see e.g. [3,18]. The construction of a hierarchical decomposition for the discontinuous Galerkin FE spaces is neither obvious nor unique. To fit the classical HB construction techniques in the nonconforming case, we search for a hierarchical decomposition of the fine grid degrees of freedom, such that one part corresponds to the degrees of freedom of the coarse grid problem. Such kind of aggregation-based hierarchical decompositions were recently studied in [5,6] for the case of Crouzeix–Raviart nonconforming FEs.

Let $A^{(0)}\mathbf{u} = \mathbf{b}$ be the algebraic formulation of our problem where $A^{(0)}$ is a symmetric positive definite graph-Laplacian, corresponding to the finest discretization $\mathcal{T}_0$. Consider a sequence of nested meshes (triangulations) $\mathcal{T}_m \subset \mathcal{T}_{m-1} \subset \cdots \subset \mathcal{T}_0$ of the domain Ω, the spaces of piece-wise constant functions $\mathcal{V}^{(m)} \subset \mathcal{V}^{(m-1)} \subset \cdots \subset \mathcal{V}^{(0)}$, the spaces of degrees of freedom $\mathbb{V}^{(m)}, \mathbb{V}^{(m-1)}, \ldots, \mathbb{V}^{(0)}$, and the number of degrees of freedom $n_m < n_{m-1} < \cdots < n_0$. Further, introduce the graph-Laplacian associated with each triangulation level, $A^{(m)}, A^{(m-1)}, \ldots, A^{(0)}$, with $(A^{(s)}u, v)_{\ell_2} = \left\langle h_{\mathcal{E}}^{-1}[u],\ [v]\right\rangle_{\mathcal{E}\cup\mathcal{E}_D}$ for $u, v \in \mathcal{V}^{(s)}$, $s = m, \ldots, 0$.

We denote by $\chi^{(k)} = \{\chi_i^{(k)}\}_{i=1}^{n_k}$ the set of standard piece-wise constant basis functions on level k and by $\widehat{\chi}^{(k)} = \{\widehat{\chi}_i^{(k)}\}_{i=1}^{n_k}$ the set of properly defined HB. The hierarchical basis $\widehat{\chi}^{(k)}$ is determined by a nonsingular transformation matrix $J^{(k)}$, i.e., $\widehat{\chi}^{(k)} = J^{(k)}\chi^{(k)}$. Then the hierarchical basis stiffness matrix $\widehat{A}^{(k)}$ and hierarchical basis spaces of degrees of freedom $\mathbb{V}^k$ are expressed as follows:

$$\widehat{\mathbb{V}}^{(k)} = J^{(k)^{-T}}\mathbb{V}^{(k)} \quad \text{and} \quad \widehat{A}^{(k)} = J^{(k)}A^{(k)}J^{(k)^T}. \tag{4.1}$$

On each level k the matrix $\widehat{A}^{(k)}$ is partitioned into a two-by-two block form:

$$\widehat{A}^{(k)} = \begin{bmatrix} \widehat{A}_{11}^{(k)} & \widehat{A}_{12}^{(k)} \\ \widehat{A}_{21}^{(k)} & \widehat{A}_{22}^{(k)} \end{bmatrix} \begin{matrix} \}n_k - n_{k+1} \\ \}n_{k+1} \end{matrix}, \tag{4.2}$$

where n_{k+1} is the dimension of the space $\mathbb{V}^{(k+1)}$. Regarding splitting (4.2) we make the following assumption:

Assumption 4.1. The hierarchical basis is locally constructed so that the transformation matrix is sparse. Moreover, the following relations hold:

$$\widehat{A}_{22}^{(k)} = A^{(k+1)}, \quad \kappa\left(\widehat{A}_{11}^{(k)}\right) = O(1). \tag{4.3}$$

Obviously splitting (4.2) generates a splitting in the space $\mathbb{V}^{(k)}$ into two subspaces in the following manner: if $\mathbf{v} = J^{(k)^T}\widehat{\mathbf{v}}$ such that $\widehat{\mathbf{v}} = (\widehat{\mathbf{v}}_1^T, \widehat{\mathbf{v}}_2^T)^T \in \widehat{\mathbb{V}}^{(k)}$ where $\widehat{\mathbf{v}}_2 \in \widehat{\mathbb{V}}^{(k+1)}$, then this gives the splitting $\mathbb{V}^{(k)} = \mathbb{V}_1^{(k)} + \mathbb{V}_2^{(k)}$ where

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 \in \mathbb{V}^{(k)} \text{ with } \mathbf{v}_1 = J^{(k)^T} \begin{bmatrix} \widehat{\mathbf{v}}_1 \\ 0 \end{bmatrix} \in \mathbb{V}_1^{(k)}, \quad \mathbf{v}_2 = J^{(k)^T} \begin{bmatrix} 0 \\ \widehat{\mathbf{v}}_2 \end{bmatrix} \in \mathbb{V}_2^{(k)}. \tag{4.4}$$

Since the matrix $A^{(k)}$ is symmetric and positive definite it generates an inner product and geometry in $\mathbb{V}^{(k)}$. The ideal case of splitting (4.4) would be when the vectors $\mathbf{v}_1, \mathbf{v}_2$ are orthogonal in the $A^{(k)}$-inner product. In any case, between these spaces $\mathbb{V}_1^{(k)}$ and $\mathbb{V}_2^{(k)}$ there is an angle in the $A^{(k)}$-inner product. The cosine of the angle is defined by the constant $\gamma^{(k)}$ in the strengthened CBS inequality:

$$(A^{(k)}\mathbf{v}_1, \mathbf{v}_2) \leqslant \gamma^{(k)} \sqrt{(A^{(k)}\mathbf{v}_1, \mathbf{v}_1)}\sqrt{(A^{(k)}\mathbf{v}_2, \mathbf{v}_2)}, \quad \mathbf{v}_1 \in \mathbb{V}_1^{(k)}, \ \ \mathbf{v}_2 \in \mathbb{V}_2^{(k)}.$$

Later in the next section this inequality will be given a different, but equivalent form, which is more convenient for estimating $\gamma^{(k)}$.

The following assumption on the constant $\gamma^{(k)}$ plays an important role in the construction of hierarchical preconditioners:

Assumption 4.2. There is an absolute constant γ such that the following inequality is valid for all $k \geqslant 0$:

$$\gamma^{(k)} \leqslant \gamma < 1.$$

We will analyze the AMLI generalization of the multiplicative two-level method, corresponding to the introduced hierarchical setting. AMLI was originally proposed by Axelsson and Vassilevski for the case of conforming linear FEs, see [3,18].

Algorithm 4.3 (*AMLI method*).

$C^{(m)} = A^{(m)}$;

for $k = 0, 1, \ldots, m-1$

$$C^{(k)} = J^{(k)^{-1}} \begin{bmatrix} \widehat{C}_{11}^{(k)} & 0 \\ \widehat{A}_{21}^{(k)} & \tilde{A}^{(k+1)} \end{bmatrix} \begin{bmatrix} I & \widehat{C}_{11}^{(k)^{-1}} \widehat{A}_{12}^{(k)} \\ 0 & I \end{bmatrix} J^{(k)^{-T}},$$

where the blocks $\widehat{C}_{11}^{(k)}$ *are symmetric positive definite approximations of* $\widehat{A}_{11}^{(k)}$, *and the Schur complement approximation is stabilized by*

$$\tilde{A}^{(k+1)^{-1}} = \left[I - p_\beta\left(C^{(k+1)^{-1}} A^{(k+1)}\right)\right] A^{(k+1)^{-1}}.$$

The acceleration polynomial is explicitly defined by

$$p_\beta(t) = \frac{1 + T_\beta\left(\dfrac{1+\alpha-2t}{1-\alpha}\right)}{1 + T_\beta\left(\dfrac{1+\alpha}{1-\alpha}\right)},$$

where $\alpha \in (0, 1)$ is a properly chosen parameter, and T_β stands for the Chebyshev polynomial of degree β with L^∞-norm 1 on $(-1, 1)$.

The following theorem is a straightforward reformulation of the basic result from [3].

Theorem 4.4. *Let Assumptions* 4.1 *and* 4.2 *hold and let the integer* β *satisfy*

$$\frac{1}{\sqrt{1-\gamma^2}} < \beta < \rho,$$

where $\rho = \max n_k/n_{k+1}$. *Then there exists* $\alpha \in (0, 1)$, *such that the AMLI preconditioner* $C^{(0)}$ *defined in* (4.3) *has optimal condition number* $\kappa\left(C^{(0)^{-1}} A^{(0)}\right) = O(1)$, *and the total computational complexity is* $O(n_0)$.

Remark 4.5. Explicit formulas for the AMLI parameter α are given in [3] where the considered acceleration polynomials are of degree $\beta = 2$ and $\beta = 3$.

The constant in the strengthened CBS inequality (CBS constant) $\gamma^{(k)}$ is a quantitative characterization of the HB. The remaining part of the paper is devoted to the construction of hierarchical splittings for the case of class of matrices represented by graph-Laplacian that satisfy Assumptions 4.1 and 4.2.

5. On the local estimates of the CBS constant

Let $\widehat{\mathbb{V}}^{(k)} = \widehat{\mathbb{V}}_1^{(k)} \times \widehat{\mathbb{V}}_2^{(k)}$ be the partitioning corresponding to the block two-by-two presentation (4.2) of the hierarchical basis stiffness matrix $\widehat{A}^{(k)}$. More appropriate for computation of the CBS constant is the following formula:

$$\gamma^{(k)} = \sup_{\mathbf{v}_i \in \widehat{\mathbb{V}}_i^{(k)}, i=1,2} \frac{\mathbf{v}_1{}^T \widehat{A}_{12}^{(k)} \mathbf{v}_2}{\sqrt{\mathbf{v}_1{}^T \widehat{A}_{11}^{(k)} \mathbf{v}_1}\sqrt{\mathbf{v}_2{}^T \widehat{A}_{22}^{(k)} \mathbf{v}_2}} = \sup_{\mathbf{v}_2 \in \widehat{\mathbb{V}}_2^{(k)}} \sqrt{\frac{\mathbf{v}_2^T \widehat{A}_{21}^{(k)} \left(\widehat{A}_{11}^{(k)}\right)^{-1} \widehat{A}_{12}^{(k)} \mathbf{v}_2}{\mathbf{v}_2^T \widehat{A}_{22}^{(k)} \mathbf{v}_2}}. \tag{5.1}$$

Now, let us assume that

$$\widehat{A}^{(k)} = \sum_{e \in \mathcal{F}} \widehat{A}_e^{(k)}, \quad \mathbf{v} = \sum_{e \in \mathcal{F}} \mathbf{v}_e,$$

where $\widehat{A}_e^{(k)}$ are symmetric positive semidefinite local matrices, $\mathcal{F}$ is some set of indices, and the summation is understood as assembling. The global hierarchical basis splitting naturally induces the block two-by-two presentation of the local matrix $\widehat{A}_e^{(k)}$, namely,

$$\widehat{A}_e^{(k)} = \begin{bmatrix} \widehat{A}_{e:11}^{(k)} & \widehat{A}_{e:12}^{(k)} \\ \widehat{A}_{e:21}^{(k)} & \widehat{A}_{e:22}^{(k)} \end{bmatrix}, \quad \mathbf{v}_e = \begin{bmatrix} \mathbf{v}_{e,1} \\ \mathbf{v}_{e,2} \end{bmatrix}. \tag{5.2}$$

Let $\widehat{\mathbb{V}}_e^{(k)}$ be the restriction of $\widehat{\mathbb{V}}^{(k)}$, corresponding to the local matrix $\widehat{A}_e^{(k)}$, and let $\widehat{\mathbb{V}}_e^{(k)} = \widehat{\mathbb{V}}_{e:1}^{(k)} \times \widehat{\mathbb{V}}_{e:2}^{(k)}$ be the partitioning corresponding to (5.2).

Lemma 5.1. *Assume that for all* $\mathbf{w} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} \in ker(\widehat{A}_e^{(k)})$, $\mathbf{v}_1 \in \widehat{\mathbb{V}}_{e:1}^{(k)}$, $\mathbf{v}_2 \in \widehat{\mathbb{V}}_{e:2}^{(k)}$, *it holds that* $\mathbf{v}_2 \in ker(\widehat{A}_{e:22}^{(k)})$. *Then the local CBS constant* $\gamma_e^{(k)}$ *is determined by*

$$\gamma_e^{(k)} = \sup_{\mathbf{v}_2 \in \widehat{\mathbb{V}}_{e:2}^{(k)} \setminus ker(\widehat{A}_{e:22}^{(k)})} \sqrt{\frac{\mathbf{v}_2^T \widehat{A}_{e:21}^{(k)} \left(\widehat{A}_{e:11}^{(k)}\right)^{-1} \widehat{A}_{e:12}^{(k)} \mathbf{v}_2}{\mathbf{v}_2^T \widehat{A}_{e:22}^{(k)} \mathbf{v}_2}} < 1, \tag{5.3}$$

and the following estimate holds:

$$\gamma^{(k)} \leqslant \max_{e \in \mathcal{F}} \gamma_e^{(k)}. \tag{5.4}$$

Proof. The assumption of the lemma is necessary condition for the correctness of (5.3), see e.g. [2,12].

Now, let $\mathbf{v}_i \in \widehat{\mathbb{V}}_i^{(k)}$, and let $\mathbf{v}_{e:i} \in \widehat{\mathbb{V}}_{e:i}^{(k)}$ be the restrictions, corresponding to the local matrices $\widehat{A}_e^{(k)}$, $i = 1, 2$. Then

$$\begin{aligned}
\left|\mathbf{v}_1^T \widehat{A}_{12}^{(k)} \mathbf{v}_2\right| &= \left|\sum_{e \in \mathcal{F}} \mathbf{v}_{e:1}^T \widehat{A}_{e:12}^{(k)} \mathbf{v}_{e:2}\right| \leqslant \sum_{e \in \mathcal{F}} \left|\mathbf{v}_{e:1}^T \widehat{A}_{e:12}^{(k)} \mathbf{v}_{e:2}\right| \\
&\leqslant \sum_{e \in \mathcal{F}} \gamma_e^{(k)} \sqrt{\mathbf{v}_{e:1}^T \widehat{A}_{e:11}^{(k)} \mathbf{v}_{e:1}} \sqrt{\mathbf{v}_{e:2}^T \widehat{A}_{e:22}^{(k)} \mathbf{v}_{e:2}} \\
&\leqslant \max_{e \in \mathcal{F}} \gamma_e^{(k)} \sum_{e \in \mathcal{F}} \sqrt{\mathbf{v}_{e:1}^T \widehat{A}_{e:11}^{(k)} \mathbf{v}_{e:1}} \sqrt{\mathbf{v}_{e:2}^T \widehat{A}_{e:22}^{(k)} \mathbf{v}_{e:2}} \\
&\leqslant \max_{e \in \mathcal{F}} \gamma_e^{(k)} \sqrt{\sum_{e \in \mathcal{F}} \mathbf{v}_{e:1}^T \widehat{A}_{e:11}^{(k)} \mathbf{v}_{e:1}} \sqrt{\sum_{e \in \mathcal{F}} \mathbf{v}_{e:2}^T \widehat{A}_{e:22}^{(k)} \mathbf{v}_{e:2}} \\
&\leqslant \max_{e \in \mathcal{F}} \gamma_e^{(k)} \sqrt{\mathbf{v}_1^T \widehat{A}_{11}^{(k)} \mathbf{v}_1} \sqrt{\mathbf{v}_2^T \widehat{A}_{22}^{(k)} \mathbf{v}_2}
\end{aligned}$$

which completes the proof. $\square$

Remark 5.2. The obtained result is a straightforward generalization of the known estimate for the standard finite element method, where the local matrices are the element stiffness matrices.

6. Estimates of the CBS constant for graph-Laplacian

Let us consider two consecutive discretizations $\mathcal{T}_k \subset \mathcal{T}_{k-1}$. In what follows we will derive uniform estimates of the CBS constant based on properly introduced construction of hierarchical basis and related decomposition of the graph-Laplacian:

$$A^{(k)} = \sum_{e \in \mathcal{E}} A_e^{(k)}, \qquad \widehat{A}^{(k)} = \sum_{e \in \mathcal{E}} \widehat{A}_e^{(k)},$$

as a sum of local matrices associated with the set of edges $\mathcal{E}$ of the coarser grid $\mathcal{T}_k$.

6.1. Mesh of triangles

Let us assume that the coarsest mesh $\mathcal{T}_m$ consists of triangles only, and each refined mesh is obtained by dividing the current triangle into four congruent triangles connecting the midpoints of its sides. Following the numbering from Fig. 2, we introduce the local matrix $A_e^{(k)}$ in the form

$$A_e^{(k)} = \left[\begin{array}{cccc|cccc}
1 & -t & \dfrac{t-1}{2} & \dfrac{t-1}{2} & & & & \\
-t & t & & & & & & \\
\dfrac{t-1}{2} & & 1+\dfrac{1-t}{2} & & & & -1 & \\
\dfrac{t-1}{2} & & & 1+\dfrac{1-t}{2} & & & & -1 \\
\hline
 & & & & 1 & -t & \dfrac{t-1}{2} & \dfrac{t-1}{2} \\
 & & & & -t & t & & \\
 & & -1 & & \dfrac{t-1}{2} & & 1+\dfrac{1-t}{2} & \\
 & & & -1 & \dfrac{t-1}{2} & & & 1+\dfrac{1-t}{2}
\end{array}\right]. \tag{6.1}$$

This edge matrix is also associated with the macroelement $E = T_1 + T_2$ of the two adjacent triangles from $\mathcal{T}_k$ with a common side e. The role of the weight parameter $t \in (0, 1)$ is correctly to distribute the contribution of the links between the interior nodes. For example, the couple (1,2) has a weight t here, but will appear also with a weight of $\dfrac{t-1}{2}$ in the local matrices associated with the rest two sides of the current triangle, so that the total contribution to have the right weight of one.

It is natural to introduce the hierarchical basis locally with respect to the triangles from $\mathcal{T}_k$. Let us consider the macroelement T_1 and the set of standard piece-wise constant basis functions $\chi_{T_1}^{(k)} = \{\chi_{T_1:i}^{(k)}\}_{i=1}^4$. We introduce the related hierarchical basis $\widehat{\chi}_{T_1}^{(k)} = \{\widehat{\chi}_{T_1:i}^{(k)}\}_{i=1}^4$ in the form

$$\widehat{\chi}_{T_1:1}^{(k)} = \chi_{T_1:1}^{(k)} + p\chi_{T_1:2}^{(k)} + q\chi_{T_1:3}^{(k)} + q\chi_{T_1:4}^{(k)},$$

$$\widehat{\chi}_{T_1:2}^{(k)} = \chi_{T_1:1}^{(k)} + q\chi_{T_1:2}^{(k)} + p\chi_{T_1:3}^{(k)} + q\chi_{T_1:4}^{(k)},$$

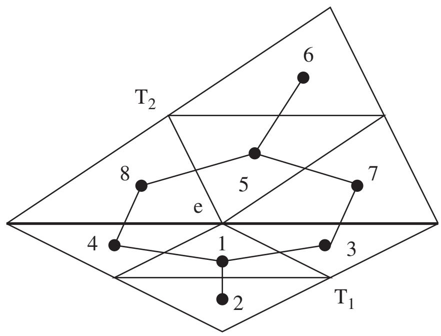

Fig. 2. Macroelement of two adjacent triangles from $\mathcal{T}_k$.

$$\widehat{\chi}^{(k)}_{T_1:3} = \chi^{(k)}_{T_1:1} + q\chi^{(k)}_{T_1:2} + q\chi^{(k)}_{T_1:3} + p\chi^{(k)}_{T_1:4},$$

$$\widehat{\chi}^{(k)}_{T_1:4} = r\left(\chi^{(k)}_{T_1:1} + \chi^{(k)}_{T_1:2} + \chi^{(k)}_{T_1:3} + \chi^{(k)}_{T_1:4}\right), \tag{6.2}$$

where p, q are parameters to be determined later, and r is the corresponding scaling factor. Then the assembled transformation matrix $J_e^{(k)}$ is as follows:

$$J_e^{(k)} = \begin{bmatrix} 1 & p & q & q & & & & \\ 1 & q & p & q & & & & \\ 1 & q & q & p & & & & \\ & & & & 1 & p & q & q \\ & & & & 1 & q & p & q \\ & & & & 1 & q & q & p \\ r & r & r & r & & & & \\ & & & & r & r & r & r \end{bmatrix} \tag{6.3}$$

and

$$\widehat{A}_e^{(k)} = J_e^{(k)} A_e^{(k)} J_e^{(k)T} = \begin{bmatrix} \widehat{A}^{(k)}_{e:11} & \widehat{A}^{(k)}_{e:12} \\ \widehat{A}^{(k)}_{e:21} & \widehat{A}^{(k)}_{e:22} \end{bmatrix}.$$

Lemma 6.1. *Consider the hierarchical basis* (6.2) *for nested meshes of triangles. Then*

$$\widehat{A}^{(k)}_{22} = A^{(k+1)} \quad \textit{if and only if} \quad r = \frac{\sqrt{2}}{2}.$$

Proof. The definition of the last two terms in the local hierarchical basis ensures that $\widehat{A}^{(k)}_{e:22}$ has row sums/column sums equal to zero. Then, the equivalent statement

$$\widehat{A}^{(k)}_{e:22} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

simply follows from the equalities

$$\widehat{A}^{(k)}_{e:22}(1,1) = r^2 \sum_{i,j=1}^{4} A_e^{(k)}(i,j) = 2r^2, \quad \widehat{A}_e^{(k)}(2,2) = r^2 \sum_{i,j=5}^{8} A_e^{(k)}(i,j) = 2r^2. \qquad \square$$

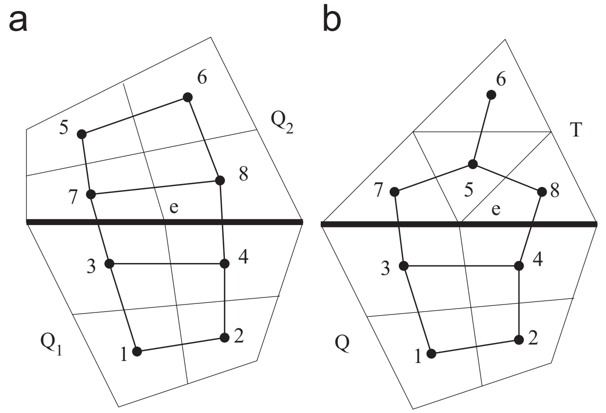

Fig. 3. (a) Macroelement of two adjacent quadrilaterals of the mesh $\mathcal{T}_k$. (b) Macroelement of adjacent triangle and quadrilateral of $\mathcal{T}_k$.

Now, it is readily seen from (5.3) that $(\gamma_e^{(k)})^2 = 1 - \lambda$ where λ is the eigenvalue (which is unique in this particular case) of the eigenproblem

$$\widehat{S}_e^{(k)}\mathbf{v} = \lambda \widehat{A}_{e:22}^{(k)}\mathbf{v}, \quad \mathbf{v} \neq \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where $\widehat{S}_e^{(k)} = \widehat{A}_{e:22}^{(k)} - \widehat{A}_{e:21}^{(k)}(\widehat{A}_{e:11}^{(k)})^{-1}\widehat{A}_{e:12}^{(k)}$.

Lemma 6.2. *Consider the hierarchical splitting* (6.1), (6.3) *with parameters* $p = 1$, $q = -0.5$, *and* $t = 0.5$. *Then the following estimate holds uniformly with respect to the refinement level* k:

$$\gamma^2 \leqslant \gamma_e^2 = \gamma_{TT}^2 = \tfrac{16}{25}. \tag{6.4}$$

Proof. The construction of the hierarchical basis and all related matrices are independent of the particular edge $e \in \mathcal{E}$ and of the current refinement level. Then, the estimate of the local CBS constant follows straightforwardly by simple computations with fixed numbers. Here, γ_{TT} indicates that the interface edge is always between two triangles. □

Remark 6.3. Varying the parameters (p, q, t) we can get a family of hierarchical splittings. For example, the parameter set $p = -1$, $q = 0$, and $t = \frac{1}{3}$ corresponds to $\gamma_e^2 = \frac{9}{13}$ which leads to the condition number estimate of the related multiplicative two-level method, $\kappa < \frac{13}{4}$. The latter result is derived by different arguments in [16].

6.2. Mesh of quadrilaterals

We assume here that the coarsest mesh $\mathcal{T}_m$ consists of quadrilaterals only, and each next refinement is obtained by dividing the current element into four new quadrilaterals as illustrated in Fig. 3(a). Following the setting of the previous subsection and the node numbering from Fig. 3, we introduce the new local matrix $A_e^{(k)}$ in the form

$$A_e^{(k)} = \left[\begin{array}{cccc|cccc} \frac{1}{2} & -s & s-\frac{1}{2} & & & & & \\ -s & \frac{1}{2} & & s-\frac{1}{2} & & & & \\ s-\frac{1}{2} & & \frac{3}{2} & -s & & & -1 & \\ & s-\frac{1}{2} & -s & \frac{3}{2} & & & & -1 \\ \hline & & & & \frac{1}{2} & -s & s-\frac{1}{2} & \\ & & & & -s & \frac{1}{2} & & s-\frac{1}{2} \\ & & -1 & & s-\frac{1}{2} & & \frac{3}{2} & -s \\ & & & -1 & & s-\frac{1}{2} & -s & \frac{3}{2} \end{array}\right]. \tag{6.5}$$

The weight parameter $s \in (0, 1)$ is again responsible for the correct distribution of the contribution of the links between the interior nodes of each quadrilateral macroelements Q_i, see Fig. 3(a).

The hierarchical basis is now introduced locally with respect to the quadrilaterals from $\mathcal{T}_k$. If we consider the macroelement Q_1, then the set of standard piece-wise constant basis functions is $\chi_{T_1}^{(k)} = \{\chi_{T_1:i}^{(k)}\}_{i=1}^4$, and the related hierarchical basis $\widehat{\chi}_{T_1}^{(k)} = \{\widehat{\chi}_{T_1:i}^{(k)}\}_{i=1}^4$ is introduced in the form

$$\begin{aligned} \widehat{\chi}_{T_1:1}^{(k)} &= (\chi_{T_1:1}^{(k)} + \chi_{T_1:2}^{(k)}) - (\chi_{T_1:3}^{(k)} + \chi_{T_1:4}^{(k)}), \\ \widehat{\chi}_{T_1:2}^{(k)} &= (\chi_{T_1:1}^{(k)} + \chi_{T_1:3}^{(k)}) - (\chi_{T_1:2}^{(k)} + \chi_{T_1:4}^{(k)}), \\ \widehat{\chi}_{T_1:3}^{(k)} &= (\chi_{T_1:1}^{(k)} + \chi_{T_1:4}^{(k)}) - (\chi_{T_1:2}^{(k)} + \chi_{T_1:3}^{(k)}), \\ \widehat{\chi}_{T_1:4}^{(k)} &= r(\chi_{T_1:1}^{(k)} + \chi_{T_1:2}^{(k)} + \chi_{T_1:3}^{(k)} + \chi_{T_1:4}^{(k)}), \end{aligned} \tag{6.6}$$

where r is again the corresponding scaling factor. Then the assembled transformation matrix $J_e^{(k)}$ reads as

$$J_e^{(k)} = \begin{bmatrix} 1 & 1 & -1 & -1 & & & & \\ 1 & -1 & 1 & -1 & & & & \\ 1 & -1 & -1 & 1 & & & & \\ & & & & 1 & 1 & -1 & -1 \\ & & & & 1 & -1 & 1 & -1 \\ & & & & 1 & -1 & -1 & 1 \\ r & r & r & r & & & & \\ & & & & r & r & r & r \end{bmatrix}. \tag{6.7}$$

We follow the local analysis scheme from the previous subsection and get the next two lemmas.

Lemma 6.4. *Consider the hierarchical basis* (6.6) *for nested meshes of quadrilaterals. Then* $\widehat{A}_{22}^{(k)} = A^{(k+1)}$ *if and only if* $r = \frac{\sqrt{2}}{2}$.

Lemma 6.5. *The estimate*

$$\gamma^2 \leqslant \gamma_e^2 = \gamma_{QQ}^2 \to \tfrac{1}{2} \tag{6.8}$$

holds uniformly with respect to the refinement level k *for the hierarchical splitting* (6.6) *with positive weight parameter* $s \to 0^+$.

Proof. The straightforward computations lead to the following expression for the Schur complement:

$$S_e = \frac{1-2s}{2(1-s)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

and therefore

$$\gamma_{QQ}^2 = 1 - \lambda = 1 - \frac{1-2s}{2(1-s)} = \frac{1}{2(1-s)}$$

which completes the proof. □

Here, γ_{QQ} indicates that the interface edge is always between two quadrilaterals.

Remark 6.6. The result from Lemma 6.5 is sympathetically equivalent to the condition number estimate of the related multiplicative two-level method, $\kappa < 2$. Applying a different technique, the later estimate is obtained in [16] for quadrilateral meshes of arbitrary space dimension.

6.3. Mesh of quadrilaterals and triangles

The general case of a coarsest mesh $\mathcal{T}_m$ consisting of quadrilaterals and triangles is considered. The refinement procedure is regular, and for the particular cases, it is the same as was considered in the previous two subsections. What remains to be analyzed is the situation, where macroelements of different kinds are adjacent as shown in Fig. 3(b). Combining the constructions from the previous two subsections and following the node numbering from Fig. 3(b), we get the local matrix $A_e^{(k)}$ in the form

$$A_e^{(k)} = \left[\begin{array}{cccc|cccc}
\frac{1}{2} & -s & s-\frac{1}{2} & & & & & \\
-s & \frac{1}{2} & & s-\frac{1}{2} & & & & \\
s-\frac{1}{2} & & \frac{3}{2} & -s & & & -1 & \\
 & s-\frac{1}{2} & -s & \frac{3}{2} & & & & -1 \\
\hline
 & & & & 1 & -t & \frac{t-1}{2} & \frac{t-1}{2} \\
 & & & & -t & t & & \\
 & & -1 & & \frac{t-1}{2} & & 1+\frac{1-t}{2} & \\
 & & & -1 & \frac{t-1}{2} & & & 1+\frac{1-t}{2}
\end{array}\right] \tag{6.9}$$

with weight parameter $s, t \in (0, 1)$. Keeping the already introduced local definitions of hierarchical bases, we write the combined transformation matrix in the form

$$J_e^{(k)} = \begin{bmatrix} 1 & 1 & -1 & -1 & & & & \\ 1 & -1 & 1 & -1 & & & & \\ 1 & -1 & -1 & 1 & & & & \\ & & & & 1 & p & q & q \\ & & & & 1 & q & p & q \\ & & & & 1 & q & q & p \\ r & r & r & r & & & & \\ & & & & r & r & r & r \end{bmatrix}. \tag{6.10}$$

Let us stress the attention on the fact that all locally introduced parameters are fixed for each particular triangle/quadrilateral macroelement from $\mathcal{T}_{k+1}$, independent of what kind of neighbors it has. In this respect it is important that $r = \frac{\sqrt{2}}{2}$ in both cases of triangles and quadrilaterals, see Lemmas 6.1 and 6.4.

Lemma 6.7. *Consider the local matrix, corresponding to the case of edge between quadrilateral and triangle, indicated below by "QT", and let* $r = \frac{\sqrt{2}}{2}$, $p = 1$, $q = -0.5$, $t = 0.5$, *and* $s \to 0^+$. *Then,*

$$\widehat{A}_{e:22}^{(k)} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

and the relation

$$\gamma_e^2 = \gamma_{QT}^2 \to \tfrac{25}{43} \tag{6.11}$$

holds uniformly with respect to the refinement level k.

Proof. Following the scheme from Lemma 6.5 we get

$$S_e = \frac{18 - 36s}{43 - 68s} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

and therefore

$$\gamma_{QT}^2 = 1 - \lambda = 1 - \frac{18 - 36s}{43 - 68s} = \frac{25 - 32s}{43 - 68s}$$

which completes the proof. □

The next two theorems summarize the results of Lemmas 6.2, 6.5, and 6.7.

Theorem 6.8. *Consider the hierarchical splitting of the graph-Laplacian, corresponding to the general case of nested meshes, where the coarsest one* $\mathcal{T}_m$ *consists of quadrilaterals and triangles.*
(a) *Then* $\widehat{A}_{22}^{(k)} = A^{(k+1)}$ *if and only if* $r = \frac{\sqrt{2}}{2}$.
(b) *If* $p = 1$, $q = -0.5$, $t = 0.5$, *and* $0 < s \leqslant \frac{35}{16} \approx 2.19$, *then,*

$$\gamma^2 \leqslant \max\{\gamma_{TT}^2, \gamma_{QQ}^2, \gamma_{QT}^2\} = \tfrac{16}{25} \quad \textit{for all } k,\ 0 \leqslant k \leqslant m. \tag{6.12}$$

Theorem 6.9. *Let the parameters of the hierarchical splitting of the graph-Laplacian satisfy conditions* (a) *and* (b) *of Theorem* 6.8. *Then the related AMLI algorithm with acceleration polynomial of degree $\beta \in \{2, 3\}$ has optimal condition number and the total computational complexity is $O(n_0)$.*

Proof. The statement follows directly from Theorems 4.4 and 6.8, taking into account that $\rho=4$ and

$$\gamma^2 \leqslant \frac{16}{25} < \frac{3}{4}. \qquad \square$$

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